Derivation of the Lorentz transformations

First, we need some definitions. We will define an inertial reference frame as a coordinate system associated with an observer moving at some constant velocity. For simplicity we will assume that this observer is at the origin, *i.e.* $x = y = z = 0$. In this coordinate system there is also a time variable which measures the elapsed time as measured on a clock at rest in this coordinate system.

Consider two frames, S and S', such that an observer in S sees the S' frame as moving in the direction of positive x with a speed v. Prior to special relativity one transformed between these frames via the Galilean transformations:

$$\begin{array}{ll} t' = t & \\ x' = x - vt & x = x' + vt' \\ y' = y & z' = z \end{array} \tag{1}$$

We saw earlier that the speed of light is required to be preserved between frames. Clearly the Galilean transformations do not preserve the speed of light, a light ray moving with speed c in the positive x direction in frame S will appear in frame S' to be moving with speed $c - v$ in the positive x' direction.

What we want is a relation between the S and S' frames which preserves the speed of light. Consider a spherically symmetric expanding wave which starts at the origin when the frames are overlaid so that $x = x' = y = y' = z = z' = t = t' = 0$. In frame S, at a given time t it is found on the spherical shell given by:

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{2}$$

and in the frame S' it is on the shell given by:

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \tag{3}$$

Now, since the frame S' is moving in the x direction relative to S it is clear by symmetry that $y' = y$ and $z' = z$. Also, the equation we get must not change if the origin of time or space is arbitrarily defined to be somewhere else, so x' and t' must be linear functions of x and t. The general form of this is:

$$x' = Ax + Bt \tag{4}$$

$$t' = Dx + Et \tag{5}$$

where A, B, D, and E are some constants.

Now, a point at rest in S' at $x' = 0$ must appear in S to be moving with speed $+v$, so it follows immediately that $-B/A = v$. We can rewrite equation 4:

$$x' = A(x - vt) \tag{6}$$

We now substitute equation 6 into equation 3:

$$\{A(x - vt)\}^2 + y^2 + z^2 - c^2(Dx + Et)^2 = 0 \tag{7}$$

Next, perform a bit of algebra on this. After some thrashing, we produce:

$$(A^2 - c^2D^2)x^2 + y^2 + z^2 + (A^2v^2 - E^2c^2)t^2 - 2(A^2v + DEc^2)xt = 0 \tag{8}$$

This equation must be identical with equation 2. Comparing terms from the two expressions we obtain the following:

$$\begin{aligned} &A^2 - c^2D^2 = 1 \\ &A^2v^2 - E^2c^2 = -c^2 \\ &A^2v + DEc^2 = 0 \end{aligned} \tag{9}$$

We can solve these equations for A, D, and E, and can discard extraneous solutions by requiring that the valid solution has the x and x' pointing in the same direction, and t and t' both increase as time passes into the future. This gives us:

$$\begin{aligned} &A = (1 - \tfrac{v^2}{c^2})^{-\frac{1}{2}} \\ &B = -vA \\ &D = -\tfrac{v}{c^2}(1 - \tfrac{v^2}{c^2})^{-\frac{1}{2}} \\ &E = (1 - \tfrac{v^2}{c^2})^{-\frac{1}{2}} \end{aligned} \tag{10}$$

If we now adopt the notation $\gamma = (1 - v^2/c^2)^{-1/2}$, we can write the Lorentz transformations as they usually appear:

$$\begin{array}{ll} x' = \gamma(x - vt) & x = \gamma(x' + vt') \\ y' = y & y = y' \\ z' = z & z = z' \\ t' = \gamma(t - \frac{vx}{c^2}) & t = \gamma(t' + \frac{vx'}{c^2}) \end{array} \tag{11}$$

Of course, the Galilean transformations in Eq. 1 make intuitive sense, the Lorentz transformations seem ridiculous. Notice that the Lorentz transformations reduce to the Galilean transformations in the limit of small velocities. Expanding γ in a McLauren series in v/c produces:

$$\begin{array}{ll} x' = (x - vt)(1 + \frac{v^2}{2c^2}) & x = (x' + vt')(1 + \frac{v^2}{2c^2}) \\ y' = y & y = y' \\ z' = z & z = z' \\ t' = (t - \frac{x}{c}\frac{v}{c})(1 + \frac{v^2}{2c^2}) & t = (t' + \frac{x}{c}\frac{v}{c})(1 + \frac{v^2}{2c^2}) \end{array} \tag{12}$$

For small values of v/c the Lorentz transformations reduce to the Galilean transformations. The departure is proportional to $(\frac{v}{c})^2$ for the spatial coordinate. The departure for the time coordinate depends a bit on the system in question. If Δx is zero in one frame then the departure of the time coordinate from the Galilean transformation value is again proportional to the square of v/c.